

Recovering Variable Order Differential Operators with Regular Singularities on Graphs.

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Abstract. We study inverse spectral problems for ordinary differential equations with regular singularities on compact star-type graphs when differential equations have different orders on different edges. As the main spectral characteristics we introduce and study the so-called Weyl-type matrices which are generalizations of the Weyl function for the classical Sturm-Liouville operator. We provide a procedure for constructing the solution of the inverse problem and prove its uniqueness.

Key words: geometrical graphs, differential operators, regular singularities, inverse spectral problems

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1. Introduction. We study inverse spectral problems for variable order differential equations with regular singularities on compact star-type graphs. More precisely, differential equations have different orders on different edges. Boundary value problems on graphs (spatial networks, trees) often appear in natural sciences and engineering (see [1-4]). Differential equations of variable orders on graphs arise in various problems in mathematics as well as in applications. In particular, we mention transverse oscillation problems for such structures as cable-stayed bridges, masts with cable supports and others.

Inverse spectral problems consist in recovering operators from their spectral characteristics. We pay attention to the most important nonlinear inverse problems of recovering coefficients of differential equations (potentials) provided that the structure of the graph is known a priori.

For *second-order* differential operators on compact graphs inverse spectral problems have been studied fairly completely in [5-10] and other works. Inverse problems for *higher-order* differential operators on graphs were investigated in [11-12]. We note that inverse spectral problems for second-order and for higher-order ordinary differential operators on *an interval* have been studied by many authors (see the monographs [13-18] and the references therein). Arbitrary order differential operators on an interval with regular singularities were considered in [19-22]. Variable order differential operators without singularities were investigated in [23-24]. Variable order differential operators on graphs with regular singularities have not been studied yet.

In this paper we study the inverse spectral problem for variable order differential operators with regular singularities on compact star-type graphs. As the main spectral characteristics in this paper we introduce and study the so-called Weyl-type matrices which are generalizations of the Weyl function (m-function) for the classical Sturm-Liouville operator (see [25]), of the Weyl matrix for higher-order differential operators on an interval introduced in [17-18], and generalizations of the Weyl-type matrices for higher-order differential operators on graphs (see [11-12]). We show that the specification of the Weyl-type matrices uniquely determines the coefficients of the differential equation on the graph, and we provide a constructive procedure for the solution of the inverse problem from the given Weyl-type matrices. For studying this inverse problem we develop the method of spectral mappings [17-18]. We also essentially use ideas from [19] on differential equations with regular singularities. The obtained results are natural generalizations of the well-known results on inverse problems for differential operators on an interval and on graphs.

2. Weyl-type matrices. Consider a compact star-type graph T in \mathbf{R}^w with the set of vertices $V = \{v_0, \dots, v_p\}$ and the set of edges $\mathcal{E} = \{e_1, \dots, e_p\}$, where v_1, \dots, v_p are the boundary vertices, v_0 is the internal vertex, and $e_j = [v_j, v_0]$, $e_1 \cap \dots \cap e_p = \{v_0\}$. Let l_j be the length of the edge e_j . Each edge $e_j \in \mathcal{E}$ is parameterized by the parameter $x_j \in [0, l_j]$ such that $x_j = 0$ corresponds to the boundary vertices v_1, \dots, v_p , and $x_j = l_j$ corresponds to

the internal vertex v_0 . An integrable function Y on T may be represented as $Y = \{y_j\}_{j=\overline{1,p}}$, where the function $y_j(x_j)$ is defined on the edge e_j .

Let n_j , $j = \overline{1,p}$, be positive integers such that $n_1 \geq n_2 \geq \dots \geq n_p \geq 2$. Consider the differential equations on T :

$$y_j^{(n_j)}(x_j) + \sum_{\mu=0}^{n_j-2} \left(\frac{\nu_{\mu j}}{x_j^{n_j-\mu}} + q_{\mu j}(x_j) \right) y_j^{(\mu)}(x_j) = \lambda y_j(x_j), \quad x_j \in (0, l_j), \quad j = \overline{1,p}, \quad (1)$$

where λ is the spectral parameter, $q_{\mu j}(x_j)$ are complex-valued integrable functions. We call $q_j = \{q_{\mu j}\}_{\mu=\overline{0, n_j-2}}$ the potential on the edge e_j , and we call $q = \{q_j\}_{j=\overline{1,p}}$ the potential on the graph T . Let $\{\xi_{kj}\}_{k=\overline{1, n_j}}$ be the roots of the characteristic polynomial

$$\delta_j(\xi) = \sum_{\mu=0}^{n_j} \nu_{\mu j} \prod_{k=0}^{\mu-1} (\xi - k), \quad \nu_{n_j, j} := 1, \quad \nu_{n_j-1, j} := 0.$$

For definiteness, we assume that $\xi_{kj} - \xi_{mj} \neq sn_j$, $s \in \mathbf{Z}$, $\operatorname{Re} \xi_{1j} < \dots < \operatorname{Re} \xi_{n_j, j}$, $\xi_{kj} \neq \overline{0, n_j - 3}$ (other cases require minor modifications). We set $\theta_j := n_j - 1 - \operatorname{Re}(\xi_{n_j, j} - \xi_{1j})$, and assume that the functions $q_{\mu j}^{(\nu)}(x_j)$, $\nu = \overline{0, \mu - 1}$, are absolutely continuous, and $q_{\mu j}^{(\mu)}(x_j) x_j^{\theta_j} \in L(0, l_j)$.

Fix $j = \overline{1,p}$. Let $\lambda = \rho_j^{n_j}$, $\varepsilon_{kj} = \exp(2\pi i k/n_j)$, $k = \overline{0, n_j - 1}$. It is known that the ρ_j -plane can be partitioned into sectors $S_{j\xi}$ of angle $\frac{\pi}{n_j}$ $\left(\arg \rho_j \in \left(\frac{\xi\pi}{n_j}, \frac{(\xi+1)\pi}{n_j} \right), \quad \xi = \overline{-n_j, n_j - 1} \right)$ in which the roots $R_{j1}, R_{j2}, \dots, R_{j, n_j}$ of the equation $R^{n_j} - 1 = 0$ can be numbered in such a way that

$$\operatorname{Re}(\rho_j R_{j1}) < \operatorname{Re}(\rho_j R_{j2}) < \dots < \operatorname{Re}(\rho_j R_{j, n_j}), \quad \rho_j \in S_{j\xi}. \quad (2)$$

Clearly, $R_{jk} = \varepsilon_{\eta_{jk}}$, where $\eta_{j1}, \dots, \eta_{j, n_j}$ is a permutation of the numbers $0, 1, \dots, n_j - 1$, depending on the sector. Let us agree that

$$\rho_j^\mu = \exp(\mu(\ln |\rho_j| + i \arg \rho_j)), \quad \arg \rho_j \in (-\pi, \pi], \quad R_{jk}^\mu = \exp(2\pi i \mu \eta_{jk}/n_j).$$

Let the numbers c_{kj0} , $k = \overline{1, n_j}$, be such that

$$\prod_{k=1}^{n_j} c_{kj0} = \left(\det[\xi_{kj}^{\nu-1}]_{k, \nu=\overline{1, n_j}} \right)^{-1}.$$

Then the functions

$$C_{kj}(x_j, \lambda) = x_j^{\xi_{kj}} \sum_{\mu=0}^{\infty} c_{kj\mu} (\rho_j x_j)^{n_j \mu}, \quad c_{kj\mu} = c_{kj0} \left(\prod_{s=1}^{\mu} \delta_j(\xi_{kj} + s n_j) \right)^{-1},$$

are solutions of the differential equation in the case when $q_{\mu j}(x_j) \equiv 0$, $\mu = \overline{0, n_j - 2}$. Moreover, $\det[C_{kj}^{(\nu-1)}(x_j, \lambda)]_{k, \nu=\overline{1, n_j}} \equiv 1$. Denote $\rho^* = \max_{j=\overline{1,p}} \left(2n_j \max_{\mu=\overline{0, n_j-2}} \|q_{\mu j}\|_{L(0, l_j)} \right)$. In [19] we constructed special fundamental systems of solutions $\{S_{kj}(x_j, \lambda)\}_{k=\overline{1, n_j}}$ and $\{E_{kj}(x_j, \rho_j)\}_{k=\overline{1, n_j}}$ of equation (1) on the edge e_j , possessing the following properties.

1) For each $x_j \in (0, l_j]$, the functions $S_{kj}^{(\nu)}(x_j, \lambda)$, $\nu = \overline{0, n_j - 1}$, are entire in λ . For each fixed λ , and $x_j \rightarrow 0$,

$$S_{kj}(x_j, \lambda) \sim c_{kj0} x_j^{\xi_{kj}}, \quad (S_{kj}(x_j, \lambda) - C_{kj}(x_j, \lambda)) x_j^{-\xi_{kj}} = o(x_j^{\xi_{n_j, j} - \xi_{1j}}).$$

Moreover, $\det[S_{kj}^{(\nu-1)}(x_j, \lambda)]_{k, \nu=\overline{1, n_j}} \equiv 1$, and $|S_{kj}^{(\nu)}(x_j, \lambda)| \leq C |x_j^{\xi_{kj} - \nu}|$, $|\rho_j| x_j \leq 1$. Here and below, we shall denote by the same symbol C various positive constants in the estimates independent of λ and x_j .

2) For each $x_j > 0$ and for each sector $S_{j\xi}$ with property (2), the functions $E_{kj}^{(\nu)}(x_j, \rho_j)$, $\nu = \overline{0, n_j - 1}$, are regular with respect to $\rho_j \in S_{j\xi}$, $|\rho_j| > \rho^*$, and continuous for $\rho_j \in \overline{S_{j\xi}}$, $|\rho_j| \geq \rho^*$. Moreover,

$$|E_{kj}^{(\nu)}(x_j, \rho_j)(\rho_j R_{jk})^{-\nu} \exp(-\rho_j R_{jk} x_j) - 1| \leq C(|\rho_j| x_j), \quad \rho_j \in \overline{S_{j\xi}}, \quad |\rho_j| x_j \geq 1.$$

3) The relation

$$E_{kj}(x_j, \rho_j) = \sum_{\mu=1}^{n_j} b_{kj\mu}(\rho_j) S_{\mu j}(x_j, \lambda), \quad (3)$$

holds, where

$$b_{kj\mu}(\rho_j) = b_{\mu j}^0 R_{jk}^{\xi_{\mu j}} \rho_j^{\xi_{\mu j}} [1], \quad b_{\mu j}^0 \neq 0, \quad \rho_j \in \overline{S_{j\xi}}, \quad \rho_j \rightarrow \infty, \quad (4)$$

$$\prod_{\mu=1}^{n_j} b_{\mu j}^0 = \det[R_{jk}^{\nu-1}]_{k, \nu=\overline{1, n_j}} \left(\det[R_{jk}^{\xi_{\mu j}}]_{k, \mu=\overline{1, n_j}} \right)^{-1},$$

where $[1] = 1 + O(\rho^{-1})$.

Note that the asymptotical formula (4) is the most important and nontrivial property of these solutions. This property allows one to study both direct and inverse problems for arbitrary order differential operators with regular singularities (see [20-22]).

Consider the linear forms

$$U_{j\nu}(y_j) = \sum_{\mu=0}^{\nu} \gamma_{j\nu\mu} y_j^{(\mu)}(l_j), \quad j = \overline{1, p}, \quad \nu = \overline{0, n_j - 1},$$

where $\gamma_{j\nu\mu}$ are complex numbers, $\gamma_{j\nu} := \gamma_{j\nu\nu} \neq 0$. The linear forms $U_{j\nu}$ will be used in matching conditions at the internal vertex v_0 for boundary value problems and for the corresponding special solutions of equation (1).

Denote $\langle n \rangle := (|n| + n)/2$, i.e. $\langle n \rangle = n$ for $n \geq 0$, and $\langle n \rangle = 0$ for $n \leq 0$. Fix $s = \overline{1, p}$, $k = \overline{1, n_s - 1}$. Let $\Psi_{sk} = \{\psi_{skj}\}_{j=\overline{1, p}}$ be solutions of equation (1) on the graph T under the boundary conditions

$$\psi_{sk s}(x_s, \lambda) \sim c_{ks0} x_s^{\xi_{ks}}, \quad x_s \rightarrow 0, \quad (5)$$

$$\psi_{skj}(x_j, \lambda) = O(x_j^{\xi_{(n_j-k-1)+2, j}}), \quad x_j \rightarrow 0, \quad j = \overline{1, p}, \quad j \neq s, \quad (6)$$

and the matching conditions at the vertex v_0 :

$$U_{1\nu}(\psi_{sk1}) = U_{j\nu}(\psi_{skj}), \quad j = \overline{2, p}, \quad \nu = \overline{0, k-1}, \quad n_j > \nu + 1, \quad (7)$$

$$\sum_{j=1, n_j > \nu}^p U_{j\nu}(\psi_{skj}) = 0, \quad \nu = \overline{k, n_s - 1}. \quad (8)$$

In particular, if $n_j > k$, then condition (6) takes the form

$$\psi_{skj}(x_j, \lambda) = O(x_j^{\xi_{n_j-k+1, j}}), \quad x_j \rightarrow 0, \quad j = \overline{1, p}, \quad j \neq s,$$

and if $n_j \leq k$, then condition (6) takes the form

$$\psi_{skj}(x_j, \lambda) = O(x_j^{\xi_{2j}}), \quad x_j \rightarrow 0, \quad j = \overline{1, p}, \quad j \neq s.$$

Matching conditions (7)-(8) are generalizations of classical matching conditions for higher-order differential operators on graphs, and matching conditions for variable order differential operators

on graphs [23-24]. The function Ψ_{sk} is called the Weyl-type solution of order k with respect to the boundary vertex v_s . Define additionally $\psi_{sns}(x_s, \lambda) := S_{ns}(x_s, \lambda)$.

Using the fundamental system of solutions $\{S_{\mu j}(x_j, \lambda)\}$ on the edge e_j , one can write

$$\psi_{skj}(x_j, \lambda) = \sum_{\mu=1}^{n_j} M_{skj\mu}(\lambda) S_{\mu j}(x_j, \lambda), \quad j = \overline{1, p}, \quad k = \overline{1, n_s - 1}, \quad (9)$$

where the coefficients $M_{skj\mu}(\lambda)$ do not depend on x_j . It follows from (9) and the boundary condition (5) for the Weyl-type solutions that

$$\psi_{sk s}(x_s, \lambda) = S_{ks}(x_s, \lambda) + \sum_{\mu=k+1}^{n_s} M_{sk\mu}(\lambda) S_{\mu s}(x_s, \lambda), \quad M_{sk\mu}(\lambda) := M_{sk s\mu}(\lambda). \quad (10)$$

We introduce the matrices $M_s(\lambda)$, $s = \overline{1, p}$, as follows:

$$M_s(\lambda) = [M_{sk\mu}(\lambda)]_{k, \mu = \overline{1, n_s}}, \quad M_{sk\mu}(\lambda) := \delta_{k\mu} \quad \text{for } k \geq \nu.$$

The matrix $M_s(\lambda)$ is called the Weyl-type matrix with respect to the boundary vertex v_s . The inverse problem is formulated as follows. Fix $w = \overline{2, p}$.

Inverse problem 1. Given $\{M_s(\lambda)\}$, $s = \overline{1, p} \setminus w$, construct q on T .

We note that the notion of the Weyl-type matrices M_s is a generalization of the notion of the Weyl function (m-function) for the classical Sturm-Liouville operator ([15, 25]) and is a generalization of the notion of Weyl matrices introduced in [11, 12, 17, 18, 20] for higher-order differential operators on an interval and on graphs. Thus, Inverse Problem 1 is a generalization of the well-known inverse problems for differential operators on an interval and on graphs.

We also note that in Inverse problem 1 we do not need to specify all matrices $M_s(\lambda)$, $s = \overline{1, p}$; one of them can be omitted. This last fact was first noticed in [6], where the inverse problem was solved for the Sturm-Liouville operators on an arbitrary tree.

In section 3 properties of the Weyl-type solutions and the Weyl-type matrices are studied. Section 4 is devoted to the solution of auxiliary inverse problems of recovering the potential on a fixed edge. In section 5 we study Inverse Problem 1. For this inverse problem we provide a constructive procedure for the solution and prove its uniqueness.

3. Properties of spectral characteristics. Fix $s = \overline{1, p}$, $k = \overline{1, n_s - 1}$. Substituting (9) into boundary and matching conditions (5)-(8), we obtain a linear algebraic system with respect to $M_{skj\mu}(\lambda)$. Solving this system by Cramer's rule one gets $M_{skj\mu}(\lambda) = \Delta_{skj\mu}(\lambda) / \Delta_{sk}(\lambda)$, where the functions $\Delta_{skj\mu}(\lambda)$ and $\Delta_{sk}(\lambda)$ are entire in λ . Thus, the functions $M_{skj\mu}(\lambda)$ are meromorphic in λ , and consequently, the Weyl-type solutions and the Weyl-type matrices are meromorphic in λ . In particular,

$$M_{sk\mu}(\lambda) = \frac{\Delta_{sk\mu}(\lambda)}{\Delta_{sk}(\lambda)}, \quad k \leq \mu,$$

where $\Delta_{sk\mu}(\lambda) := \Delta_{sk s\mu}(\lambda)$. We note that the function $\Delta_{sk}(\lambda)$ is the characteristic function for the boundary value problem L_{sk} for equation (1) under the conditions

$$y_s(x_s) = O(x_s^{\xi_{k+1, s}}), \quad x_s \rightarrow 0, \quad y_j(x_j) = O(x_j^{\xi_{(n_j - k - 1) + 2, j}}), \quad x_j \rightarrow 0, \quad j = \overline{1, p}, \quad j \neq s,$$

$$U_{1\nu}(y_1) = U_{j\nu}(y_j), \quad j = \overline{2, p}, \quad \nu = \overline{0, k - 1}, \quad n_j > \nu + 1,$$

$$\sum_{j=1, n_j > \nu}^p U_{j\nu}(y_j) = 0, \quad \nu = \overline{k, n_s - 1}.$$

Zeros of $\Delta_{sk}(\lambda)$ coincide with the eigenvalues of L_{sk} . Denote

$$\Omega_{jk} = \det[R_{jl}^{\xi_{\mu j}}]_{l, \mu = \overline{1, k}}, \quad \Omega_{j0} = 1, \quad \omega_{jk} := \frac{\Omega_{j, k-1}}{\Omega_{jk}}, \quad k = \overline{1, n_j}.$$

Lemma 1. Fix $j = \overline{1, p}$, and fix a sector $S_{j\xi}$ with property (2).

1) Let $k = \overline{1, n_j - 1}$, and let $y_j(x_j, \lambda)$ be a solution of equation (1) on the edge e_j under the condition

$$y_j(x_j, \lambda) = O(x_j^{\xi_{k+1, j}}), \quad x_j \rightarrow 0. \quad (10)$$

Then for $x_j \in (0, l_j]$, $\nu = \overline{0, n_j - 1}$, $\rho_j \in S_{j\xi}$, $|\rho_j| \rightarrow \infty$,

$$y_j^{(\nu)}(x_j, \lambda) = \sum_{\mu=k+1}^{n_j} A_{\mu j}(\rho_j)(\rho_j R_{j\mu})^\nu \exp(\rho_j R_{j\mu} x_j)[1], \quad (11)$$

where the coefficients $A_{\mu j}(\rho_j)$ do not depend on x_j . Here and below we assume that $\arg \rho_j = \text{const}$, when $|\rho_j| \rightarrow \infty$.

2) Let $k = \overline{1, n_j}$, and let $y_j(x_j, \lambda)$ be a solution of equation (1) on the edge e_j under the condition

$$y_j(x_j, \lambda) \sim c_{kj0} x_j^{\xi_{kj}}, \quad x_j \rightarrow 0. \quad (12)$$

Then for $x_j \in (0, l_j]$, $\nu = \overline{0, n_j - 1}$, $\rho_j \in S_{j\xi}$, $|\rho_j| \rightarrow \infty$,

$$y_j^{(\nu)}(x_j, \lambda) = \frac{\omega_{jk}}{\rho_j^{\xi_{kj}}} (\rho_j R_{jk})^\nu \exp(\rho_j R_{jk} x_j)[1] + \sum_{\mu=k+1}^{n_j} B_{\mu j}(\rho_j)(\rho_j R_{j\mu})^\nu \exp(\rho_j R_{j\mu} x_j)[1], \quad (13)$$

where the coefficients $B_{\mu j}(\rho_j)$ do not depend on x_j .

Proof. It follows from (10) that

$$y_j(x_j, \lambda) = \sum_{\mu=k+1}^{n_j} a_{\mu j}(\lambda) S_{\mu j}(x_j, \lambda). \quad (14)$$

Using the fundamental system of solutions $\{E_{kj}(x_j, \rho_j)\}_{k=\overline{1, n_j}}$, one can write

$$y_j(x_j, \lambda) = \sum_{m=1}^{n_j} A_{mj}(\rho_j) E_{mj}(x_j, \rho_j). \quad (15)$$

By virtue of (3), we calculate

$$y_j(x_j, \lambda) = \sum_{m=1}^{n_j} A_{mj}(\rho_j) \sum_{\mu=1}^{n_j} b_{mj\mu}(\rho_j) S_{\mu j}(x_j, \lambda) = \sum_{\mu=1}^{n_j} S_{\mu j}(x_j, \lambda) \sum_{m=1}^{n_j} A_{mj}(\rho_j) b_{mj\mu}(\rho_j).$$

Comparing this relation with (14), we obtain

$$\sum_{m=1}^{n_j} A_{mj}(\rho_j) b_{mj\mu}(\rho_j) = 0, \quad \mu = \overline{1, k}. \quad (16)$$

We consider (16) as a linear algebraic system with respect to $A_j(\rho_j), A_{2j}(\rho_j), \dots, A_{kj}(\rho_j)$. Solving this system by Cramer's rule and taking (4) into account we get

$$A_{mj}(\rho_j) = \sum_{\mu=k+1}^{n_j} (\alpha_{m\mu j} + O(\rho_j^{-1})) A_{\mu j}(\rho_j), \quad m = \overline{1, k}, \quad (17)$$

where $\alpha_{m\mu j}$ are constants. Substituting (17) into (15) and using (2) we arrive at (11). Relations (13) are proved analogously by using (12) instead of (10). \square

Now we are going to study the asymptotic behavior of the Weyl-type solutions.

Lemma 2. *Fix $s = \overline{1, p}$, $k = \overline{1, n_s}$, and fix a sector $S_{s\xi}$ with property (2). For $x_s \in (0, l_s)$, $\nu = \overline{0, n_s - 1}$, the following asymptotic formula holds*

$$\psi_{sks}^{(\nu)}(x_s, \lambda) = \frac{\omega_{sk}}{\rho_s^{\xi_{ks}}} (\rho_s R_{sk})^\nu \exp(\rho_s R_{sk} x_s) [1], \quad \rho_s \in S_{s\xi}, \quad |\rho_s| \rightarrow \infty. \quad (18)$$

Proof. For $k = n_s$, (18) follows from Lemma 1. Fix $s = \overline{1, p}$, $k = \overline{1, n_s - 1}$. Using Lemma 1 and boundary conditions for Ψ_{sk} we get the following asymptotic formulae for $|\lambda| \rightarrow \infty$, inside the corresponding sectors:

$$\psi_{sks}^{(\nu)}(x_s, \lambda) = \frac{\omega_{sk}}{\rho_s^{\xi_{ks}}} (\rho_s R_{sk})^\nu \exp(\rho_s R_{sk} x_s) [1] + \sum_{\mu=k+1}^{n_s} A_{\mu s}^{sk}(\rho_s) (\rho_s R_{s\mu})^\nu \exp(\rho_s R_{s\mu} x_s) [1], \quad x_s \in (0, l_s], \quad (19)$$

$$\psi_{skj}^{(\nu)}(x_j, \lambda) = \sum_{\mu=\langle n_j-k+1 \rangle + 2}^{n_j} A_{\mu j}^{sk}(\rho_j) (\rho_j R_{j\mu})^\nu \exp(\rho_j R_{j\mu} x_j) [1], \quad j = \overline{1, p} \setminus s, \quad x_j \in (0, l_j]. \quad (20)$$

Substituting (19)-(20) into matching conditions (7)-(8) for Ψ_{sk} , we obtain the linear algebraic system with respect to $A_{\mu j}^{sk}(\rho_j)$. Solving this system by Cramer's rule, we obtain in particular,

$$A_{\mu s}^{sk}(\rho_s) = O(\rho_s^{-\xi_{ks}} \exp(\rho_s (R_{sk} - R_{s\mu}) l_s)). \quad (21)$$

Substituting (21) into (19) we arrive at (18). \square

It follows from the proof of Lemma 2 that one can also get the asymptotics for $\psi_{skj}^{(\nu)}(x_j, \lambda)$, $j \neq s$; but for our purposes only (18) is needed.

4. Auxiliary inverse problems. In this section we consider auxiliary inverse problems of recovering differential operator on each fixed edge. Fix $s = \overline{1, p}$, and consider the following auxiliary inverse problem on the edge e_s .

IP(s). Given the matrix M_s , construct the potential q_s on the edge e_s .

In this inverse problem we construct the potential only on the edge e_s , but the Weyl-type matrix M_s brings a global information from the whole graph. In other words, this problem is not a local inverse problem related only to the edge e_s .

Theorem 1. *Fix $s = \overline{1, p}$. The specification of the Weyl-type matrix M_s uniquely determines the potential q_s on the edge e_s .*

We omit the proof since it is similar to that in [18, Ch.2]. Moreover, using the method of spectral mappings and the asymptotics (18) for the Weyl-type solutions, one can get a constructive procedure for the solution of the inverse problem $IP(s)$. It can be obtained by the same arguments as for n -th order differential operators on a finite interval (see [18, Ch.2] for details). Note that like in [18], the nonlinear inverse problem $IP(s)$ is reduced to the solution of a linear equation in the corresponding Banach space of sequences.

Fix $j = \overline{1, p}$. Let $\varphi_{jk}(x_j, \lambda)$, $k = \overline{1, n_j}$, be solutions of equation (1) on the edge e_j under the conditions

$$\varphi_{kj}^{(\nu-1)}(l_j, \lambda) = \delta_{k\nu}, \quad \nu = \overline{1, k}, \quad \varphi_{kj}(x_j, \lambda) = O(x_j^{\xi_{n_j-k+1, j}}), \quad x_j \rightarrow 0.$$

We introduce the matrix $m_j(\lambda) = [m_{jk\nu}(\lambda)]_{k, \nu=\overline{1, n_j}}$, where $m_{jk\nu}(\lambda) := \varphi_{jk}^{(\nu-1)}(l_j, \lambda)$. The matrix $m_j(\lambda)$ is called the Weyl-type matrix with respect to the internal vertex v_0 and the edge e_j .

IP[j]. Given the matrix m_j , construct q_j on the edge e_j .

This inverse problem is the classical one, since it is the inverse problem of recovering a higher-order differential equation on a finite interval from its Weyl-type matrix. This inverse problem has been solved in [18], where the uniqueness theorem for this inverse problem is proved. Moreover, in [18] an algorithm for the solution of the inverse problem $IP[j]$ is given, and necessary and sufficient conditions for the solvability of this inverse problem are provided.

5. Solution of Inverse Problem 1. In this section we obtain a constructive procedure for the solution of Inverse problem 1 and prove its uniqueness. First we prove an auxiliary assertion.

Lemma 3. Fix $j = \overline{1, p}$. Then for each fixed $s = \overline{1, p} \setminus j$,

$$m_{j1\nu}(\lambda) = \frac{\psi_{s1j}^{(\nu-1)}(l_j, \lambda)}{\psi_{s1j}(l_j, \lambda)}, \quad \nu = \overline{2, n_j}, \quad (22)$$

$$m_{jk\nu}(\lambda) = \frac{\det[\psi_{s\mu j}(l_j, \lambda), \dots, \psi_{s\mu j}^{(k-2)}(l_j, \lambda), \psi_{s\mu j}^{(\nu-1)}(l_j, \lambda)]_{\mu=\overline{1, k}}}{\det[\psi_{s\mu j}^{(\xi-1)}(l_j, \lambda)]_{\xi, \mu=\overline{1, k}}}, \quad 2 \leq k < \nu \leq n_j. \quad (23)$$

Proof. Denote

$$w_{js}(x_j, \lambda) := \frac{\psi_{s1j}(x_j, \lambda)}{\psi_{s1j}(l_j, \lambda)}.$$

The function $w_{js}(x_j, \lambda)$ is a solution of equation (1) on the edge e_j , and $w_{js}(l_j, \lambda) = 1$. Moreover, by virtue of the boundary conditions on Ψ_{s1} , one has $w_{js}(x_j, \lambda) = O(x_j^{\xi_{n_j, j}})$, $x_j \rightarrow 0$. Hence, $w_{js}(x_j, \lambda) \equiv \varphi_{1j}(x_j, \lambda)$, i.e.

$$\varphi_{1j}(x_j, \lambda) = \frac{\psi_{s1j}(x_j, \lambda)}{\psi_{s1j}(l_j, \lambda)}. \quad (24)$$

Similarly, we calculate

$$\varphi_{kj}(x_j, \lambda) = \frac{\det[\psi_{s\mu j}(l_j, \lambda), \dots, \psi_{s\mu j}^{(k-2)}(l_j, \lambda), \psi_{s\mu j}(x_j, \lambda)]_{\mu=\overline{1, k}}}{\det[\psi_{s\mu j}^{(\xi-1)}(l_j, \lambda)]_{\xi, \mu=\overline{1, k}}}, \quad k = \overline{2, n_j - 1}. \quad (25)$$

Since $m_{jk\nu}(\lambda) = \varphi_{kj}^{(\nu-1)}(l_j, \lambda)$, it follows from (24) that (22) holds. Similarly, (23) follows from (25). \square

Now we are going to obtain a constructive procedure for the solution of Inverse problem 1. Our plan is the following.

Step 1. Let the Weyl-type matrices $\{M_s(\lambda)\}$, $s = \overline{1, p} \setminus w$, be given. Solving the inverse problem $IP(s)$ for each fixed $s = \overline{1, p} \setminus w$, we find the potentials q_s on the edges e_s , $s = \overline{1, p} \setminus w$.

Step 2. Using the knowledge of the potential on the edges e_s , $s = \overline{1, p} \setminus w$, we construct the Weyl-type matrix $m_w(\lambda)$.

Step 3. Solving the inverse problem $IP[w]$, we find the potential q_w on e_w .

Steps 1 and 3 have been already studied in Section 3. It remains to fulfil Step 2. For this purpose it is convenient to divide differential equations into m groups with equal orders. More precisely, let $\omega_1 > \omega_2 > \dots > \omega_m > \omega_{m+1} = 1$, $n_{p_{j-1}+1} = \dots = n_{p_j} := \omega_j$, $j = \overline{1, m}$, $0 = p_0 < p_1 < \dots < p_m = p$. Take N such that $p_N = w$.

Suppose that Step 1 is already made, and we found the potentials q_s , $s = \overline{1, p} \setminus p_N$, on the edges e_s , $s = \overline{1, p} \setminus p_N$. Then we calculate the functions $S_{kj}(x_j, \lambda)$, $j = \overline{1, p} \setminus p_N$; here $k = \overline{1, \omega_i}$ for $j = \overline{p_{i-1} + 1, p_i}$.

Fix $s = \overline{1, p_1}$ (if $N > 1$), and $s = \overline{1, p_1 - 1}$ (if $N = 1$). All calculations below will be made for this fixed s . Our goal now is to construct the Weyl-type matrix $m_{p_N}(\lambda)$. According to (22)-(23), in order to construct $m_{p_N}(\lambda)$ we have to calculate the functions

$$\psi_{skp_N}^{(\nu)}(l_{p_N}, \lambda), \quad k = \overline{1, \omega_N - 1}, \quad \nu = \overline{0, \omega_N - 1}. \quad (26)$$

We will find the functions (26) by the following steps.

1) Using (10) we construct the functions

$$\psi_{sks}^{(\nu)}(l_s, \lambda), \quad k = \overline{1, \omega_N - 1}, \quad \nu = \overline{0, \omega_1 - 1}, \quad (27)$$

by the formula

$$\psi_{sks}^{(\nu)}(l_s, \lambda) = S_{ks}^{(\nu)}(l_s, \lambda) + \sum_{\mu=k+1}^{\omega_1} M_{sk\mu}(\lambda) S_{\mu s}^{(\nu)}(l_s, \lambda). \quad (28)$$

2) Consider a part of the matching conditions (7) on Ψ_{sk} . More precisely, let $\xi = \overline{N, m}$, $k = \overline{\omega_{\xi+1}, \omega_{\xi} - 1}$, $l = \overline{\xi, m}$, $j = \overline{1, p_l - 1}$. Then, in particular, (7) yields

$$U_{p_l, \nu}(\psi_{skp_l}) = U_{j\nu}(\psi_{skj}), \quad \nu = \overline{\omega_{l+1} - 1, \min(k - 1, \omega_l - 2)}. \quad (29)$$

Since the functions (27) are known, it follows from (29) that one can calculate the functions

$$\psi_{skj}^{(\nu)}(l_j, \lambda), \quad \xi = \overline{N, m}, \quad k = \overline{\omega_{\xi+1}, \omega_{\xi} - 1}, \quad l = \overline{\xi, m}, \quad j = \overline{1, p_l}, \quad \nu = \overline{\omega_{l+1} - 1, \min(k - 1, \omega_l - 2)}. \quad (30)$$

In particular we found the functions (26) for $\nu = \overline{0, k - 1}$.

3) It follows from (9) and the boundary conditions on Ψ_{sk} that

$$\psi_{skj}^{(\nu)}(l_j, \lambda) = \sum_{\mu=\max(\omega_l-k+1)}^{\omega_l} M_{skj\mu}(\lambda) C_{\mu j}^{(\nu)}(l_j, \lambda), \quad (31)$$

$$k = \overline{1, \omega_1 - 1}, \quad l = \overline{1, m}, \quad j = \overline{p_{l-1} + 1, p_l} \setminus s, \quad \nu = \overline{0, \omega_l - 1}.$$

We consider only a part of relations (31). More precisely, let $\xi = \overline{N, m}$, $k = \overline{\omega_{\xi+1}, \omega_{\xi} - 1}$, $l = \overline{1, m}$, $j = \overline{p_{l-1} + 1, p_l}$, $j \neq p_N$, $j \neq s$, $\nu = \overline{0, \min(k - 1, \omega_l - 2)}$. Then

$$\sum_{\mu=\max(\omega_l-k+1)}^{\omega_l} M_{skj\mu}(\lambda) C_{\mu j}^{(\nu)}(l_j, \lambda) = \psi_{skj}^{(\nu)}(l_j, \lambda), \quad \nu = \overline{0, \min(k - 1, \omega_l - 2)}. \quad (32)$$

For this choice of parameters, the right-hand side in (32) are known, since the functions (30) are known. Relations (32) form a linear algebraic system σ_{skj} with respect to the coefficients $M_{skj\mu}(\lambda)$. Solving the system σ_{skj} by Cramer's rule we find the functions $M_{skj\mu}(\lambda)$. Substituting them into (31), we calculate the functions

$$\psi_{skj}^{(\nu)}(l_j, \lambda), \quad k = \overline{1, \omega_N - 1}, \quad l = \overline{1, m}, \quad j = \overline{p_{l-1} + 1, p_l} \setminus p_N, \quad \nu = \overline{0, \omega_l - 1}. \quad (33)$$

Note that for $j = s$ these functions were found earlier.

4) Let us now use the generalized Kirchhoff's conditions (8) for Ψ_{sk} . Since the functions (33) are known, one can construct by (8) the functions (26) for $k = \overline{1, \omega_N - 1}$, $\nu = \overline{k, \omega_N - 1}$. Thus, the functions (26) are known for $k = \overline{1, \omega_N - 1}$, $\nu = \overline{0, \omega_N - 1}$.

Since the functions (26) are known, we construct the Weyl-type matrix $m_{p_N}(\lambda)$ via (22)-(23) for $j = p_N$. Thus, we have obtained the solution of Inverse problem 1 and proved its uniqueness, i.e. the following assertion holds.

Theorem 2. *The specification of the Weyl-type matrices $M_s(\lambda)$, $s = \overline{1, p} \setminus p_N$, uniquely determines the potential q on T . The solution of Inverse problem 1 can be obtained by the following algorithm.*

Algorithm 1. *Given the Weyl-type matrices $M_s(\lambda)$, $s = \overline{1, p} \setminus p_N$.*

1) *Find q_s , $s = \overline{1, p} \setminus p_N$, by solving the inverse problem $IP(s)$ for each fixed $s = \overline{1, p} \setminus p_N$.*

2) *Calculate $C_{kj}^{(\nu)}(l_j, \lambda)$, $j = \overline{1, p} \setminus p_N$; here $k = \overline{1, \omega_i}$, $\nu = \overline{0, \omega_i - 1}$ for $j = \overline{p_{i-1} + 1, p_i}$.*

- 3) Fix $s = \overline{1, p_1}$ (if $N > 1$), and $s = \overline{1, p_1 - 1}$ (if $N = 1$). All calculations below will be made for this fixed s . Construct the functions (27) via (28).
- 4) Calculate the functions (30) using (29).
- 5) Find the functions $M_{skj\mu}(\lambda)$, by solving the linear algebraic systems σ_{skj} .
- 6) Construct the functions (26) using (8).
- 7) Calculate the Weyl-type matrix $m_{p_N}(\lambda)$ via (22)-(23) for $j = p_N$.
- 8) Construct the potential q_{p_N} on the edge e_{p_N} by solving the inverse problem $IP[j]$ for $j = p_N$.

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